On a family of stars of maximal weight covering exactly k nodes

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March 24, 2004

Abstract

We prove the NP–hardness of the following decision problem: given a complete graph with edge weights $w_e \in \mathbb{Z}_+$, $k \in \mathbb{Z}_+$, and $w^\star \in \mathbb{Z}_+$. Does there exist an edge induced subgraph with exactly k nodes and total weight at least $w^\star$ while each connected component is a star, i.e. a tree of depth 1.

1 Introduction

The problem of finding a node cover in a graph with minimum cardinality is quite old ([NR59] for example). Just recently Plesník presented the closely related problem of finding a minimum weight edge subset covering exactly a given number k of nodes and proved its hardness in [P99]. Since negative edge weights are used in this proof it is natural to take a look at the case with only positive edge weights. Plesník again proved that this problem is polynomially solvable using an algorithm that identifies a solution covering at least k nodes. Furthermore, he could modify such a solution, if necessary, to cover exactly k nodes (see [P01] for details).

Since the proposed polynomial procedure from Plesník uses matching techniques, one might expect that the general problem, using arbitrary rational numbers, could be solved in polynomial time as well. This idea is supported by the basic stars like structure, that can be assumed for any optimal solution. For unweighted trees it has been shown in [HF$^+$04] that finding a cover of exactly k nodes with stars is easy. But we will show in this note that even the simplified problem of minimizing over negative weights, and thus maximizing over positive weights, is NP–complete for general graphs.

We assume familiarity with graph theory and computer science. Our notation is fairly standard as in [GJ79]. Additionally, a graph with at least two nodes is called a star, if there exists one node (called *center* of the star) that is adjacent with all other nodes and all those other nodes have degree 1 (they are called *outer* nodes of the star).

This simlified *node cover by stars* problem (or k–max–NCS for short) may now be stated as follows:

Instance: Complete graph $G = (V, E)$ with non negative edge weights, a positive number $k \in \mathbb{Z}_+$, and a non negative number $w \in \mathbb{Z}_+$.

Question: Does there exist a covering of k nodes by disjoint stars so that the weight of all selected edges is at least $w^\star$?

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In the remainder of this technical report we prove the hardness of this problem. For that purpose we give a reduction from *exact cover by 3–sets (X3C)* (see e.g. [GJ79]).

Instance: Set X with $|X| = 3q$ and a collection $\mathcal{C}$ of 3–element subsets of X.

Question: Does $\mathcal{C}$ contain an exact cover for X, i.e. a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $X = \bigcup_{C_j \in \mathcal{C}'} C_j$ and $|\mathcal{C}'| = q$.

2 Reduction

Theorem 1 *k–max–NCS is NP–complete.*

Proof:

It is obvious that this node cover problem is in NP.

Let $(X, \mathcal{C})$ be an instance of $X3C$. We transform this into an instance of k–max–NCS as follows.

- Create a complete graph $G = (V, E)$ with the following nodes:
  1. nodes o_i for each element $x_i \in X$
  2. two nodes c_j and u_j for each set $C_j \in \mathcal{C}$.
- The weights of the edges are of three types.
  1. For all $C_j \in \mathcal{C}$ we have $w_{(c_j,u_j)} = 2|\mathcal{C}| + 1$.
  2. For all $C_j = \{x_{j1}, x_{j2}, x_{j3}\} \in \mathcal{C}$ we put $w_{(c_j,o_{ji})} = |\mathcal{C}| + 1$ for $i = 1, 2, 3$.
  3. All other edges in the complete graph get weight 0.
- Furthermore, let $k = 5q$ and $w^\star = (5|\mathcal{C}| + 4)q$.

Clearly, this transformation can be computed in polynomial time.

Note, each set C_j corresponds to a unique star $S(C_j)$ of weight $w(S(C_j)) = 5|\mathcal{C}| + 4$ with center c_j.

Given an exact three cover $\mathcal{C}'$ we cover $5|\mathcal{C}| = 5q$ nodes by the set of stars $\{S(C_j) \mid C_j \in \mathcal{C}'\}$ with the correct weight.

Now, if $\mathcal{S}$ is a collection of stars covering $5q$ nodes with a weight of $w^\star = (5|\mathcal{C}| + 4)q$, we will show that all stars must be of type $S(C_j)$ for some $C_j \in \mathcal{C}$ implying the existence of an exact cover. For that purpose we show:

Claim 1 *If S is a star covering p nodes, then $w(S) \leq |p| \left(|\mathcal{C}| + \frac{4}{5}\right)$ with equality if and only if there exists $C_j \in \mathcal{C}$ such that $S = S(C_j)$.*

We may assume that such a star does not use any edges of weight zero. This leaves two types of stars:

stars T with center o_i: T covers $|T| + 1$ nodes. Using $|T| \leq \mathcal{C}$ we compute

$$w(T) = |T|(|\mathcal{C}| + 1) = (|T| + 1)(|\mathcal{C}| + |T| - |\mathcal{C}|) \leq (|T| + 1)|\mathcal{C}|.$$

stars S with center c_j: We may assume, that $(c_j, u_j) \in S$ as this edge is of largest weight and all other edges incident with u_j have weight zero. Let $l \in \{0, 1, 2, 3\}$ denote the number of edges of weight $(|\mathcal{C}| + 1)$ in S. Then

$$w(S) = 2|\mathcal{C}| + 1 + l(|\mathcal{C}| + 1) = (l + 2)\left(|\mathcal{C}| + \frac{l+1}{l+2}\right) \leq (l + 2)\left(|\mathcal{C}| + \frac{4}{5}\right)$$

with equality if and only if $l = 3$. □

For the minimization problem with rational weights, we were originally interested in, we derive as a corollary:

Corollary 1 *The problem k–min–NCS with rational weights is NP–complete.*

References

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