Some Results on a
Paint Shop Problem for Words

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1 Introduction

Motivated by an application in the automobile industry, we present results and conjectures on a new combinatorial problem.

We are given a word \(w = (w_1, \ldots, w_n)\) with letters \(w_i\) of an alphabet \(B\), and a color vector \(f = (f_1, \ldots, f_n)\) with colors \(f_i\) of a color set \(F\). Each \(f_i\) denotes the color of \(w_i\). Whenever \(f_i \neq f_{i+1}\), we say that we have a color change in \(f\).

Problem 1 Paint Shop Problem for Words (PPW)

Given a finite alphabet \(B\), a word \(w = (w_1, \ldots, w_n) \in B^*\), a set \(F\) of colors and a coloring \(f = (f_1, \ldots, f_n)\) of \(w\), find a permutation \(\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}\) such that \(w_{\sigma(i)} = w_i\) for \(i = 1, \ldots, n\) and the number of all color changes within \(\sigma(f) = (f_{\sigma(1)}, \ldots, f_{\sigma(n)})\) is minimized.

Given an instance \((w; f)\) of PPW, we denote the number of its color changes by \(\gamma(w)\) and the optimal (minimal) number of color changes by \(\gamma^*(w)\). Note, that the initial coloring of a word determines the reservoir of letters available in each color. Thus, we can deal with these reservoirs instead of a color vector. We denote the reservoir of letter \(i\) in color \(j\) by \(V(i, j)\).

Considering the letters \(w_i\) as car bodies, Problem 1 refers to the problem of coloring a given car body sequence in a paint shop. As each color change gives rise to substantial cost and pollution, the minimization of color changes is aspired by the automobile industry for a long time (see [1] and references therein).

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Even restricted versions of Problem 1 are \( \mathcal{NP} \)-complete. We restrict to instances of bounded size of \( F \) resp. \( B \) and show by reduction from 3SAT resp. pseudo-polynomial reduction from 3-PARTITION that even in these cases the problem remains \( \mathcal{NP} \)-complete.

**Theorem 2** \( PPW \) is \( \mathcal{NP} \)-complete for \( |F| = 2 \).

**Theorem 3** \( PPW \) is \( \mathcal{NP} \)-complete for \( |B| = 2 \).

Each instance \((w; f)\) of PPW can be solved by a dynamic program. We only have to pass through \( w \) (letter by letter from the left to the right) and record each feasible coloring up to the current position in a different state.

**Theorem 4** An instance of PPW with letters of an alphabet \( B \) and colors of a color set \( F \) can be solved with a space and time complexity of \( O(|F|n|F||B|) \).

Details on the results of this section can be found in [2].

### 3 \( k \)-regular Instances

Given a fixed integer \( k \geq 1 \), we call an instance \((w; f)\) of PPW \( k \)-regular, if \( V(i, j) = k = \frac{|w|}{|F|} \) holds for all letters \( i \) and colors \( j \). We first restrict to the case of \( k \)-regular instances of bounded size of \( B \) and give an upper bound for the value \( \gamma^*(w) \), starting with a simple lemma.

**Lemma 5** Suppose we are given a \( k \)-regular instance of PPW with \( |B| = |F| = 2 \). Then \( \gamma^*(w) \leq 2 \) holds.

We use Lemma 5 to prove Theorem 6 by induction.

**Theorem 6** Suppose we are given a \( k \)-regular instance of PPW with \( |B| = 2 \). Then \( \gamma^*(w) \leq 2(|F| - 1) \) holds.

Indeed, besides the dynamic program mentioned in Theorem 4 we know no efficient way to compute an optimal coloring (even for \( k \)-regular instances). When dealing with instances of bounded size of \( F \) instead of \( B \), we can not even show an upper bound for an optimal coloring (like in Theorem 6). Thus, we only present a conjecture for this case.

**Conjecture 7** Suppose we are given a \( k \)-regular instance of PPW with \( |F| = 2 \). Then \( \gamma^*(w) \leq |B| \) holds.
Note, that Conjecture 7 is correct for \( k = 1 \). Combining Theorem 6 and Conjecture 7 results in Conjecture 8.

**Conjecture 8** Suppose we are given a \( k \)-regular instance of PPW. Then \( \gamma^*(w) \leq |B|(|F| - 1) \) holds.

The following example proves that the bound given in Conjecture 8 is tight if the conjecture is correct.

**Example 9** Suppose we are given a \( k \)-regular instance of PPW with a color set \( F \) and an alphabet \( B = \{b_1, \ldots, b_{|F|}\} \) of the form

\[
 w = (\underbrace{b_1 \ldots b_1}_{|F|} \underbrace{b_2 \ldots b_2}_{|F|} \underbrace{\ldots}_{|F|} \underbrace{b_{|B|} \ldots b_{|B|}}_{|F|}).
\]

Then \( \gamma^*(w) = |B|(|F| - 1) \) holds.

Finally, we take a look at \( 1 \)-regular instances \((w; f)\) with \(|F| = 2\). One might expect that the natural greedy approach (when coloring \( w \) from the left to the right, keep the actual color as long as possible) produces good results. Example 10 shows that in general this is not the case.

**Example 10** Suppose we are given a \( 1 \)-regular instance of PPW with \(|F| = 2\) and \( B = \{b_1, \ldots, b_{|F|}\} \) (with \(|B| \) even) of the form

\[
 w = (\underbrace{b_1 \ldots b_{|B|/2}}_{|F|/2} \underbrace{b_{|B|/2} \ldots b_{|B|/2}}_{|F|/2} b_{|B|/2+1} b_{|B|/2+1} \underbrace{b_{|B|/2} b_{|B|/2} \ldots b_{|B|/2}}_{|F|/2-1} b_{|B|-1}).
\]

The greedy algorithm defined above colors the word \( w \) with \(|B| = 2^n = O(n)\) color changes, while the minimal number of color changes is always \( \gamma^*(w) = 3 \).

We end with an open problem.

**Problem 11** Given a \( 1 \)-regular instance \((w; f)\) of PPW with \(|F| = 2\), compute or approximate the optimal value \( \gamma^*(w) \).

**References**
